VARIETIES WHOSE FINITELY GENERATED MEMBERS ARE FREE

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Abstract. We prove that a variety of algebras whose finitely generated members are free must be definitionally equivalent to the variety of sets, the variety of pointed sets, a variety of vector spaces over a division ring, or a variety of affine vector spaces over a division ring.

1. Introduction

In this paper we address a MathOverflow question, [1], which asks for a description of the varieties where every algebra is free, as well as a description of the varieties satisfying the weaker requirement that every finitely generated algebra is free.

Steven Givant classified the varieties where every algebra is free in [4]. He proved that they are precisely those definitionally equivalent to

- the variety of sets,
- the variety of pointed sets,
- a variety of vector spaces over a division ring, or
- a variety of affine spaces over a division ring.

In this paper we use different techniques to classify the varieties where every finitely generated algebra is free. Our result is that if the finitely generated members of a variety $\mathcal{V}$ are free, then $\mathcal{V}$ must also be one of these types of varieties (sets, pointed sets, vector spaces or affine spaces). Hence, if the finitely generated algebras in $\mathcal{V}$ are free, then all algebras in $\mathcal{V}$ are free. This gives a new proof of Givant's Theorem under weaker hypotheses.

In the last section of the paper we discuss some variations on the main question. First we consider a "large rank" variation: Which varieties have the property that their finitely generated algebras of sufficiently large rank are free? That is, for which varieties $\mathcal{V}$ is there a finite number k such that every finitely generated algebra in $\mathcal{V}$ requiring more than k generators is free? We prove the theorem that a locally finite variety with this property must even have the property that all of its nonsingleton

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algebras are free, and it is essentially one of the four types of varieties discussed above. Without the assumption of local finiteness this theorem fails.

Next we examine a "small rank" variation of the main question: Is there some n such that, if all $(\leq n)$-generated algebras in a variety are free, then all finitely generated algebras in the variety are free? The answer to this is negative. We show that for each positive integer n there exist varieties in which the algebras generated by at most n elements are free, but the $(n+1)$-generated algebras are not all free.

2. Abelian and affine algebras

Please refer to [3, 5, 8] for elaboration of the introductory remarks of this section.

An algebra $\mathbf{A}$ is *abelian* if it satisfies the *term condition*, which is the assertion that if $t(\mathbf{x}, \mathbf{y})$ is a term in the language, $\mathbf{a}, \mathbf{b}, \mathbf{u}$ and $\mathbf{v}$ are tuples of elements of A, and

$$t^{\mathbf{A}}(\underline{\mathbf{a}}, \mathbf{u}) = t^{\mathbf{A}}(\underline{\mathbf{a}}, \mathbf{v}),$$

then

$$t^{\mathbf{A}}(\underline{\mathbf{b}}, \mathbf{u}) = t^{\mathbf{A}}(\underline{\mathbf{b}}, \mathbf{v}).$$

This property is the same as the property that the diagonal $\{(a, a) \mid a \in A\}$ of $\mathbf{A} \times \mathbf{A}$ is the class of a congruence.

An algebra $\mathbf{B}$ is *affine* if it is polynomially equivalent to a module. This means that there is a ring R and a left R-module structure ${}_RB$ on the universe B of $\mathbf{B}$ such that the polynomial operations of $\mathbf{B}$ coincide with the R-module polynomial operations of ${}_RB$. (A polynomial operation of an algebra $\mathbf{B}$ is an operation $p(\mathbf{x})$ obtained from a term operation by substituting constants for some of the variables, i.e. $p(\mathbf{x}) = t^{\mathbf{B}}(\mathbf{x}, \mathbf{b})$ for some term $t(\mathbf{x}, \mathbf{y})$ in the language and some tuple $\mathbf{b}$ of elements of B.)

A variety is abelian or affine if its members are. It is a fact that affine algebras and varieties are abelian, but the converse is false, e.g. unary varieties are abelian but not affine.

Abelian varieties that are not affine are poorly understood at present. If $\mathcal{V}$ is a locally finite variety that is abelian but not affine, then it can be proved that $\mathcal{V}$ contains a very "bad" or "structureless" algebra, i.e. one that is definitionally equivalent to a matrix power of a two-element set or pointed set. The procedure for proving this is to first exploit the nonaffineness assumption to construct a finite "strongly abelian" algebra $\mathbf{S} \in \mathcal{V}$, and then to examine a minimal subvariety of the variety $\mathsf{H}\,\mathsf{S}\,\mathsf{P}(\mathbf{S})$ generated by $\mathbf{S}$. The structure of such minimal subvarieties are determined by the classification theorem for minimal abelian varieties, which can be found in [9] and [12]. Namely, a minimal subvariety of a variety generated by a finite strongly abelian algebra is definitionally equivalent to a matrix power of the variety of sets or the variety of pointed sets.

These arguments fail at the very first step for varieties that are not locally finite: it is not known if the construction discussed in the preceding paragraph yields an algebra $\mathbf{S}$ that is strongly abelian. In this section we examine the construction of $\mathbf{S}$ and identify some "strongly abelian–like" properties of $\mathbf{S}$.

First, a congruence $\theta \in \mathrm{Con}(\mathbf{A})$ is *strongly abelian* if it satisfies the strong term condition, which is the assertion that if $t(\mathbf{x}, \mathbf{y})$ is a term in the language, $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{u}, \mathbf{v}$ are tuples of elements of A with $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ θ-related coordinatewise and $\mathbf{u}, \mathbf{v}$ θ-related coordinatewise, and

$$t^{\mathbf{A}}(\underline{\mathbf{a}}, \mathbf{u}) = t^{\mathbf{A}}(\underline{\mathbf{b}}, \mathbf{v}),$$

then

$$t^{\mathbf{A}}(\underline{\mathbf{c}}, \mathbf{u}) = t^{\mathbf{A}}(\underline{\mathbf{c}}, \mathbf{v}).$$

Now suppose that $\mathbf{A}$ is abelian and $\theta \in \mathrm{Con}(\mathbf{A})$ is strongly abelian. The construction we are concerned with is the following one: Let $\mathbf{A}(\theta)$ be the subalgebra of $\mathbf{A} \times \mathbf{A}$ supported by (the graph of) θ, that is, $\{(a, b) \in A \mid a \equiv_\theta b\}$. Let Δ be the congruence on $\mathbf{A}(\theta)$ generated by $D \times D$ where $D = \{(a, a) \mid a \in A\}$ is the diagonal. D is a Δ-class, because $\mathbf{A}$ is abelian. Let $\mathbf{S} = \mathbf{S}_{\mathbf{A},\theta} := \mathbf{A}(\theta)/\Delta$. Let $0 = D/\Delta \in S$.

If $\mathbf{A}$ is a finite member of an abelian variety, then it is possible to prove that the resulting algebra $\mathbf{S}$ is a strongly abelian member of the variety (meaning that all of its congruences are strongly abelian). Without finiteness we do not know how to prove this. However, we can prove the following.

Lemma 2.1. *Let $\mathcal{V}$ be an abelian variety, and suppose that θ is a nontrivial strongly abelian congruence on some $\mathbf{A} \in \mathcal{V}$. Let $\mathbf{S} = \mathbf{S}_{\mathbf{A},\theta}$ and let $0 = D/\Delta \in S$. The following are true:*

(1) $\mathbf{S}$ *has more than one element.*
(2) $\{0\}$ *is a 1-element subuniverse of* $\mathbf{S}$.
(3) $\mathbf{S}$ *has "Property P": for every n-ary polynomial $p(\mathbf{x})$ of $\mathbf{S}$ and every tuple $\mathbf{s} \in S^n$*

$$p(\mathbf{s}) = 0 \quad \textit{implies} \quad p(\mathbf{0}) = 0,$$

where $\mathbf{0} = (0, 0, \ldots, 0)$.

(4) *Whenever $t(x_1, \ldots, x_n)$ is a $\mathcal{V}$-term, and*

$$\mathcal{V} \models t(\mathbf{x}) = t(\mathbf{y})$$

where $\mathbf{x}$ and $\mathbf{y}$ are tuples of not necessarily distinct variables which differ in the ith position, then the term operation $t^{\mathbf{S}}(x_1, \ldots, x_n)$ is independent of its ith variable.

(5) $\mathbf{S}$ *has a congruence σ such that the algebra $\mathbf{S}/\sigma$ satisfies (1)–(4) of this lemma, and $\mathbf{S}/\sigma$ also has a compatible partial order $\leq$ such that $0 \leq s$ for every $s \in S/\sigma$.*

Proof. [Item (1)] Since $\mathbf{A}$ is abelian, the diagonal D is the class of a congruence on $\mathbf{A}\times\mathbf{A}$, namely the congruence generated by $D\times D$. This congruence restricts to $\mathbf{A}(\theta)$ to have D as a class. Since θ is nontrivial, it properly contains D, so the congruence Δ of $\mathbf{A}(\theta)$ generated by $D\times D$ is proper. Equivalently, $\mathbf{S}=\mathbf{A}(\theta)/\Delta$ is nontrivial.

[Item (2)] Since D is a subuniverse of $\mathbf{A}(\theta)$, $\{D/\Delta\}=\{0\}$ is a subuniverse of $\mathbf{S}$.

[Item (3)] To show that $\mathbf{S}$ has Property P, choose $p(\mathbf{x})$ and $\mathbf{s}\in S^n$ such that $p(\mathbf{s})=0$. Our goal is to show that $p(\mathbf{0})=0$.

Express $p(\mathbf{x})$ as $t^{\mathbf{S}}(\mathbf{x},\mathbf{u})$ for some term $t(\mathbf{x},\mathbf{y})$ and for some tuple $\mathbf{u}$ with coordinates in S. Also, express the coordinates s_i and u_j of the tuples $\mathbf{s}$ and $\mathbf{u}$ as $s_i=(a_i,b_i)/\Delta$ and $u_j=(v_j,w_j)/\Delta$ where $(a_i,b_i),(v_j,w_j)\in\theta$. Then $p(\mathbf{s})=0$ may be expressed as $t^{\mathbf{A}(\theta)}\big((\mathbf{a},\mathbf{b}),(\mathbf{v},\mathbf{w})\big)\in D$, or

$$t^{\mathbf{A}}(\underline{\mathbf{a}},\mathbf{v})=t^{\mathbf{A}}(\underline{\mathbf{b}},\mathbf{w}).$$

Since θ is strongly abelian, by the strong term condition we derive that

$$t^{\mathbf{A}}(\underline{\mathbf{a}},\mathbf{v})=t^{\mathbf{A}}(\underline{\mathbf{a}},\mathbf{w})$$

holds, which may be expressed as $t^{\mathbf{A}(\theta)}((\mathbf{a},\mathbf{a}),(\mathbf{v},\mathbf{w}))\in D$, or $p(\mathbf{0})=p((\mathbf{a},\mathbf{a})/\Delta)=0$.

[Item (4)] Assume for the sake of simplicity that $i=1$ in the statement of (4), that is, $\mathcal{V}\models t(x,\mathbf{w})=t(y,\mathbf{z})$. By specializing if necessary we may assume further that $w_j,z_j\in\{x,y\}$ for all j. Our goal is to show that $t^{\mathbf{S}}(x_1,\dots,x_n)$ is independent of its first variable.

Claim 2.2. *For any $s\in S$, $t^{\mathbf{S}}(s,0,0,\dots,0)=0$.*

Proof of Claim. The identity $t(x,\mathbf{w})=t(y,\mathbf{z})$ may be written symbolically as

$$t((x,y),(\mathbf{w},\mathbf{z}))\in D,$$

where (x,y) and each (w_j,z_j) belong to the set $\{(x,y),(y,x),(x,x),(y,y)\}$.

Choose $s\in S$ and represent it as $s=(a,b)/\Delta$ for some pair $(a,b)\in\theta$. Each of the pairs $(a,b),(b,a),(a,a),(b,b)$ belongs to θ, so we may substitute a's and b's for x's and y's to obtain that

$$t^{\mathbf{A}(\theta)}((a,b),(\mathbf{c},\mathbf{d}))\in D,$$

where each (c_j,d_j) is one of the elements of $\{(a,b),(b,a),(a,a),(b,b)\}$. Factoring by Δ yields

$$t^{\mathbf{S}}\big((a,b)/\Delta,(\mathbf{c},\mathbf{d})/\Delta\big)=t^{\mathbf{S}}\big(s,\underline{(\mathbf{c},\mathbf{d})/\Delta}\big)=0. \tag{2.1}$$

Now we apply Property P to the polynomial $p(\mathbf{y})=t^{\mathbf{S}}(s,\mathbf{y})$ to change the underlined values in (2.1) to 0. We obtain that $t^{\mathbf{S}}(s,\mathbf{0})=0$, as desired. ∎

Recall that $\mathbf{S} \in \mathcal{V}$ is abelian. Therefore, for arbitrary $s \in S$, we may apply the term condition to

$$t^{\mathbf{S}}(s, \underline{\mathbf{0}}) = t^{\mathbf{S}}(0, \underline{\mathbf{0}}) = 0$$

to obtain

$$t^{\mathbf{S}}(s, \underline{\mathbf{u}}) = t^{\mathbf{S}}(0, \underline{\mathbf{u}})$$

for any $\mathbf{u}$. This is what it means for $t^{\mathbf{S}}(x_1, \ldots, x_n)$ to be independent of its first variable.

[Item (5)] Let R be the reflexive compatible relation on $\mathbf{S}$ generated by $\{0\} \times S$. Hence R consists of all pairs $(p(\mathbf{0}), p(\mathbf{s}))$ where $\mathbf{s}$ is a tuple of elements of $\mathbf{S}$ and p is a polynomial of $\mathbf{S}$. Property P asserts exactly that $(x, 0) \in R$ implies $x = 0$. The transitive closure R^* of R also has this property. Therefore the symmetrization $\sigma := R^* \cap (R^*)^{\cup}$ is a congruence on $\mathbf{S}$ and $\leq := R^*/\sigma$ is a compatible partial order on the quotient $\mathbf{S}/\sigma$. This partial order contains $(\{0\} \times S)/(\sigma \times \sigma)$, so $0 \leq s$ for every $s \in S/\sigma$.

Note that $\mathbf{S}/\sigma$ satisfies all of the earlier properties. (1): The quotient $\mathbf{S}/\sigma$ is nontrivial, since $\mathbf{S}$ is nontrivial and $\{0\}$ is a singleton class of σ. (2): $\{0\}/\sigma$ is a singleton subuniverse of the quotient. (3): Property P is easily derivable from a lower bounded compatible order: $0 \leq p(\mathbf{0}) \leq p(\mathbf{s})$ for any $\mathbf{s}$, so $p(\mathbf{s}) = 0$ implies $p(\mathbf{0}) = 0$. (4): The assumption of part (4) of the Lemma statement depends on $\mathcal{V}$ only, while the conclusion is preserved when taking quotients. □

The properties that have been proved for $\mathbf{S} = \mathbf{S}_{\mathbf{A},\theta}$ and its quotient $\mathbf{S}/\sigma$ prevent $\mathcal{V}$ from being affine. For example, no nontrivial affine algebra can satisfy Property P: let $p(x) = x - s$ for some $s \in S \setminus \{0\}$. Then $p(s) = 0$ while $p(0) \neq 0$. In fact, this polynomial has no fixed points at all.

Similarly, an affine algebra has no compatible reflexive relations other than equivalence relations. If the compatible partial order in (5) was an equivalence relation, then it would be discrete. For the discrete order to have a least element 0, the underlying set could have only one element, contrary to item (1).

Also, it is not hard to show that a variety that contains an algebra $\mathbf{S}/\sigma$ satisfying the property described in item (4) cannot satisfy any nontrivial idempotent Maltsev condition, while affine varieties satisfy strong idempotent Maltsev conditions (they in fact have a Maltsev-term). These observations justify the following definition.

Definition 2.3. An algebra $\mathbf{S}$ is called an *affine obstruction* if it contains an element 0 such that conditions (1)–(4) of Lemma 2.1 hold for $\mathbf{S}$ and the variety $\mathcal{V}$ generated by $\mathbf{S}$.

Theorem 2.4. *The following are equivalent for an abelian variety $\mathcal{V}$.*

(1) *$\mathcal{V}$ is not affine.*
(2) *$\mathcal{V}$ satisfies no nontrivial idempotent Maltsev condition.*

(3) *$\mathcal{V}$ contains an algebra that has a nontrivial strongly abelian congruence.*
(4) *$\mathcal{V}$ contains an affine obstruction.*

Proof. [(1) $\Rightarrow$ (2)] (First proof.) We argue the contrapositive, so assume that $\mathcal{V}$ satisfies a nontrivial idempotent Maltsev condition. By Theorem 4.16 (2) of [8], congruence lattices of algebras in $\mathcal{V}$ omit pentagons with certain specified abelian intervals. Since $\mathcal{V}$ is abelian, all intervals in congruence lattices of members are abelian. Hence there are no pentagons in congruence lattices of members of $\mathcal{V}$, which means that $\mathcal{V}$ is congruence modular. In this context it is known that abelian varieties are affine (see [3]).

[(1) $\Rightarrow$ (2)] (Second proof.) Again we argue the contrapositive, so assume that $\mathcal{V}$ satisfies a nontrivial idempotent Maltsev condition. By Theorem 3.21 of [8], $\mathcal{V}$ has a join term. The join term acts as a semilattice operation on blocks of any rectangular tolerance of an algebra in $\mathcal{V}$. Since every algebra in $\mathcal{V}$ is abelian and there are no nontrivial abelian semilattices, it follows that rectangular tolerances in $\mathcal{V}$ are trivial. (This fact can also be deduced from Corollary 5.15 of [8].) Now by Theorem 5.25 of [8], it follows that $\mathcal{V}$ satisfies an idempotent Maltsev condition that fails in the variety of semilattices. By Theorem 4.10 of [10], $\mathcal{V}$ is affine.

[(2) $\Leftrightarrow$ (3)] This is part of Theorem 3.13 of [8].

[(3) $\Rightarrow$ (4)] If $\mathcal{V}$ contains an algebra $\mathbf{A}$ with a nontrivial strongly abelian congruence θ, then it contains $\mathbf{S} = \mathbf{S}_{\mathbf{A},\theta} := \mathbf{A}(\theta)/\Delta$, which is an affine obstruction by Lemma 2.1.

[(4) $\Rightarrow$ (1)] Here it suffices to prove that an affine obstruction for $\mathcal{V}$ prevents $\mathcal{V}$ from being affine. This was explained right after the proof of Lemma 2.1. $\square$

3. Varieties whose finitely generated members are free

In this section we investigate the class of varieties whose finitely generated members are free. This class of varieties is closed under definitional equivalence. The symbol $\mathcal{V}$ will be used only to denote some nontrivial member of this class. We shall divide our analysis of this class into two cases: the subclass of varieties with no 0-ary function symbols versus the subclass of varieties with at least one 0-ary function symbol.

We shall prove that if the finitely generated members of $\mathcal{V}$ are free, then $\mathcal{V}$ must be definitionally equivalent to the variety of sets, pointed sets, vector spaces over a division ring, or affine spaces over a division ring. It is obvious that each of these varieties has the property that its finitely generated members are free.

3.1. Varieties without constants.

First we will consider the case when $\mathcal{V}$ has no 0-ary function symbols. We may write the m-generated free algebra in $\mathcal{V}$ as $\mathbf{F}_{\mathcal{V}}(m)$, or as $\mathbf{F}_{\mathcal{V}}(X)$ for some m-element set X.

Theorem 3.1. *Assume that $\mathcal{V}$ is a nontrivial variety such that the finitely generated algebras in $\mathcal{V}$ are free. If $\mathcal{V}$ has no 0-ary function symbols, then $\mathcal{V}$ is definitionally equivalent to the variety of sets or to a variety of affine spaces over a division ring.*

Proof. If $\mathcal{V}$ has no 0-ary function symbols, then $\mathbf{F}_{\mathcal{V}}(\emptyset)$ is empty. $\mathbf{F}_{\mathcal{V}}(1)$ is the only candidate for the 1-element algebra in $\mathcal{V}$. Hence $\mathcal{V}$ is idempotent.

It follows from the standard proofs of Magari's Theorem ([2], Theorem 10.13) that every nontrivial variety has a *finitely generated* simple member. A free algebra $\mathbf{F}_{\mathcal{V}}(X)$ over $X = \{x_1, x_2, \ldots\}$ cannot be simple if $|X| > 2$, since there are noninjective homomorphisms $\varepsilon_i \colon \mathbf{F}_{\mathcal{V}}(X) \to \mathbf{F}_{\mathcal{V}}(y, z)$ defined on generators by

$$x_j \mapsto \begin{cases} y & \text{if } j = i \\ z & \text{else.} \end{cases} \tag{3.1}$$

If $\mathcal{V}$ is idempotent, then $\mathbf{F}_{\mathcal{V}}(X)$ cannot be simple for $|X| < 2$, either. Thus, in our situation $\mathbf{F}_{\mathcal{V}}(2)$ is the only candidate for a finitely generated simple member of $\mathcal{V}$.

Let $\mathcal{M}$ be a minimal subvariety of $\mathcal{V}$. $\mathcal{M}$ also must contain a finitely generated simple algebra, and $\mathbf{F}_{\mathcal{V}}(2)$ is the only one in $\mathcal{V}$ up to isomorphism, so $\mathcal{M}$ must contain (and be generated by) $\mathbf{F}_{\mathcal{V}}(2)$. Every finitely generated algebra $\mathbf{A} \in \mathcal{M}$ is finitely generated in $\mathcal{V}$, hence is free in $\mathcal{V}$, hence satisfies the universal mapping property in $\mathcal{V}$ relative to some subset $X \subseteq A$, hence satisfies the universal mapping property in $\mathcal{M}$ relative to the same subset, hence is free over the same free generating set in $\mathcal{M}$. This shows that $\mathcal{M}$ is also a variety whose finitely generated algebras are free. Also, $\mathbf{F}_{\mathcal{M}}(2) = \mathbf{F}_{\mathcal{V}}(2)$.

According to Corollary 2.10 of [7], any minimal idempotent variety, like $\mathcal{M}$, is definitionally equivalent to the variety of sets, the variety of semilattices, a variety of affine modules over a simple ring, or is congruence distributive.

The variety of semilattices does not have the property that its finitely generated members are free.

No minimal, congruence distributive, idempotent variety $\mathcal{M}$ has the property that its finitely generated members are free, as we now explain. If otherwise, then since $\mathbf{F}_{\mathcal{M}}(x, y) \times \mathbf{F}_{\mathcal{M}}(x, y)$ is finitely generated (by $\{x, y\} \times \{x, y\}$), it must be isomorphic to $\mathbf{F}_{\mathcal{M}}(m)$ for some m. Since $\mathbf{F}_{\mathcal{M}}(x, y) \times \mathbf{F}_{\mathcal{M}}(x, y)$ is not trivial or simple, we have $m > 2$. The homomorphisms $\{\varepsilon_i\}_{i=1}^{m}$ described in (3.1) (with subscript $\mathcal{M}$ in place of $\mathcal{V}$) map $\mathbf{F}_{\mathcal{M}}(m)$ onto the simple algebra $\mathbf{F}_{\mathcal{M}}(2)$, and ε_i has kernel different from that of ε_j when $i \neq j$. Thus $\mathbf{F}_{\mathcal{M}}(m)$ has at least m distinct coatoms of the form $\ker(\varepsilon_i)$ in its congruence lattice. From this it follows that $\mathbf{F}_{\mathcal{M}}(x, y) \times \mathbf{F}_{\mathcal{M}}(x, y) \cong \mathbf{F}_{\mathcal{M}}(m)$, $m > 2$, has at least 3 coatoms in its congruence lattice. But in a congruence distributive variety, the square of a simple algebra has exactly two coatoms in its congruence lattice.

Now consider the case where $\mathcal{M}$ is a variety of affine (left) modules over some ring R. One realization of $\mathbf{F}_{\mathcal{M}}(2)$ has universe R, generators $0, 1 \in R$, and term

operations of the form

$$r_1x_1 + \cdots + r_hx_h, \quad r_i \in R, \quad \sum r_i = 1.$$

Each left ideal of R induces a congruence on this algebra. Since $\mathbf{F}_{\mathcal{M}}(2)$ is simple, R can have no nontrivial proper left ideals, hence R must be a division ring.

We have thus far argued that if $\mathcal{V}$ has the property that its finitely generated members are free, and $\mathcal{M}$ is a minimal subvariety of $\mathcal{V}$, then $\mathcal{M}$ is definitionally equivalent to the variety of sets or a variety of affine modules over a division ring. We now argue that $\mathcal{V} = \mathcal{M}$. If this is not the case, then there is a finitely generated algebra in $\mathcal{V} \setminus \mathcal{M}$, which we may assume is $\mathbf{A} := \mathbf{F}_{\mathcal{V}}(m)$. By its very definition, $\mathbf{A}$ has an m-element generating set that is minimal under inclusion as a generating set. Now let $\mathbf{B}$ be the m-generated free algebra in $\mathcal{M}$. So $\mathbf{B}$ also has an m-element minimal generating set. Since $\mathbf{B} \in \mathcal{M}$, we get that $\mathbf{B} \in \mathcal{V}$, but $\mathbf{B}$ cannot be isomorphic to $\mathbf{A}$, because $\mathbf{A} \notin \mathcal{M}$. Hence $\mathbf{B} \cong \mathbf{F}_{\mathcal{V}}(n)$ for some $n \neq m$. This implies that $\mathbf{B}$ has an n-element minimal generating set as well as an m-element minimal generating set. But $\mathcal{M}$ is definitionally equivalent to the variety of sets or to a variety of affine spaces over a division ring, so it is not possible for $\mathbf{B}$ to have minimal generating sets of different cardinalities. We conclude that $\mathcal{V} = \mathcal{M}$. $\square$

3.2. **Varieties with constants.** We still assume that $\mathcal{V}$ is a nontrivial variety whose finitely generated members are free. In this subsection we also assume that $\mathcal{V}$ has 0-ary function symbols in its language. In this situation, $\mathbf{F}_{\mathcal{V}}(\emptyset)$ must be the 1-element algebra in $\mathcal{V}$, so there is only one constant up to equivalence. We will assume that there is exactly one constant in the language and use 0 to denote it. In any algebra $\mathbf{A} \in \mathcal{V}$ the set $\{0\}$ is the unique 1-element subuniverse of $\mathbf{A}$. We will refer to $0 \in \mathbf{A}$ as the *zero element* of $\mathbf{A}$.

In the situation we are in now, when $\mathbf{F}_{\mathcal{V}}(\emptyset) = \{0\}$, it is $\mathbf{F}_{\mathcal{V}}(1)$ rather than $\mathbf{F}_{\mathcal{V}}(2)$ that is the only candidate for the finitely generated simple algebra of $\mathcal{V}$. To see this, note that when m is greater than 1, then $\mathbf{F}_{\mathcal{V}}(x_1, \ldots, x_m)$ has at least three distinct kernels of homomorphisms onto $\mathbf{F}_{\mathcal{V}}(x)$, namely the kernels of the homomorphisms defined on generators by

(1) $x_1 \mapsto 0$; $x_2, \ldots, x_m \mapsto x$,
(2) $x_1 \mapsto x$; $x_2, \ldots, x_m \mapsto 0$, and
(3) $x_1, x_2, \ldots, x_m \mapsto x$.

To see that the kernels of these homomorphisms are distinct, it suffices to note that they restrict differently to the set $\{0, x_1, \ldots, x_m\} \subseteq F_{\mathcal{V}}(x_1, \ldots, x_m)$. Thus $\mathbf{F}_{\mathcal{V}}(m)$ cannot be simple when $m > 1$, nor can it be simple when $m = 0$, hence $\mathbf{F}_{\mathcal{V}}(1)$ is the finitely generated simple member of $\mathcal{V}$. This argument also shows that, if $m > 1$, then $\mathbf{F}_{\mathcal{V}}(m)$ has at least 3 coatoms in its congruence lattice. We record these observations as:

Lemma 3.2. *If $\mathcal{V}$ is a nontrivial variety with at least one* 0*-ary function symbol in its language, and all finitely generated members of $\mathcal{V}$ are free, then*

(1) $\mathbf{F}_{\mathcal{V}}(\emptyset)$ *has one element.*
(2) $\mathbf{F}_{\mathcal{V}}(1)$ *is simple.*
(3) $\mathbf{F}_{\mathcal{V}}(m)$ *has at least* 3 *distinct coatoms in its congruence lattice for every finite* $m > 1$. □

Later we will need to remember that, from part (3) of this lemma, any finitely generated, nontrivial, nonsimple member of $\mathcal{V}$ has at least 3 distinct coatoms in its congruence lattice.

Suppose that $\mathbf{A} \in \mathcal{V}$ and $a \in A \setminus \{0\}$. Then there is a homomorphism $\mathbf{F}_{\mathcal{V}}(x) \to \mathbf{A}$ mapping $x \mapsto a$, which cannot be constant (since $0 \mapsto 0$). By the simplicity of $\mathbf{F}_{\mathcal{V}}(x)$, this homomorphism must be injective. This shows that a is a free generator of the subalgebra $\langle a \rangle \leq \mathbf{A}$. We record this as:

Lemma 3.3. *If $\mathcal{V}$ is a nontrivial variety with a* 0*-ary function symbol* 0*, and all finitely generated members of $\mathcal{V}$ are free, then any nonzero element of any algebra in $\mathcal{V}$ freely generates a subalgebra isomorphic to $\mathbf{F}_{\mathcal{V}}(x)$.* □

Lemma 3.4. *If $\mathcal{V}$ is a nontrivial variety with a* 0*-ary function symbol, and all finitely generated members of $\mathcal{V}$ are free, then $\mathbf{F}_{\mathcal{V}}(x)$ is abelian.*

Proof. In this proof we will abbreviate $\mathbf{F}_{\mathcal{V}}(x)$ by $\mathbf{F}$.

Let $\mathbf{A}$ be the subalgebra of $\mathbf{F} \times \mathbf{F}$ that is generated by $(0, x)$ and $(x, 0)$. Let $\eta_1, \eta_2 \in \mathrm{Con}(\mathbf{A})$ be the restrictions to $\mathbf{A}$ of the coordinate projection kernels. Observe that the η_1-class of $0^{\mathbf{A}} = (0, 0)$ is the set $\{0\} \times F$, which is a subuniverse of $\mathbf{A}$ that supports a subalgebra isomorphic to $\mathbf{F}$; hence this subalgebra is simple. Similarly, the η_2-class of $0^{\mathbf{A}}$, $F \times \{0\}$, is the universe of a simple subalgebra of $\mathbf{A}$.

$\mathbf{A}$ is generated by $(0, x)$ and $(x, 0)$, so every class of the congruence $\mathrm{Cg}\left((0,0),(0,x)\right)$ of $\mathbf{A}$ contains an element of $F \times \{0\}$. As $\mathrm{Cg}\left((0,0),(0,x)\right)$ is contained in η_1, and each η_1-class contains exactly one element of $F \times \{0\}$, it follows that $\eta_1 = \mathrm{Cg}\left((0,0),(0,x)\right)$.

This shows that η_1 is principal, hence compact, so there is a congruence μ that is maximal among congruences strictly below η_1. $\mathbf{Con}(\mathbf{A}/\mu)$ contains a 3-element maximal chain $0 = \mu/\mu \prec \eta_1/\mu \prec 1$. We apply Lemma 3.2 (3) to $\mathbf{A}/\mu$: the algebra $\mathbf{A}/\mu$ is nontrivial, nonsimple, and a quotient of the 2-generated algebra $\mathbf{A}$, so it is finitely generated. The lemma guarantees that $\mathbf{Con}(\mathbf{A}/\mu)$ has at least 3 coatoms. The congruence η_1/μ is a coatom, but there must be at least two other coatoms, say $\alpha, \beta \in \mathbf{Con}(\mathbf{A}/\mu)$.

Since α, β and η_1/μ are pairwise incomparable congruences, and η_1/μ is an atom in $\mathbf{Con}(\mathbf{A}/\mu)$, we have $\alpha \wedge (\eta_1/\mu) = 0 = \beta \wedge (\eta_1/\mu)$. We also have

$$(\alpha \vee \beta) \wedge (\eta_1/\mu) = 1 \wedge (\eta_1/\mu) = \eta_1/\mu,$$

so the interval $[0, \eta_1/\mu]$ is a meet semidistributivity failure in $\mathbf{Con}(\mathbf{A}/\mu)$. It follows from basic properties of the commutator that η_1/μ is abelian.

Recall that $\{0\} \times F$ is a subuniverse of $\mathbf{A}$ that is an η_1-class. The congruence μ is strictly smaller than $\eta_1 = \mathrm{Cg}\left((0,0),(0,x)\right)$, so it does not contain $\{0\} \times F$ entirely within a class. Since the subuniverse supported by $\{0\} \times F$ is isomorphic to $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(1)$, and therefore simple, μ restricts trivially to this set. This implies that $(\{0\} \times F)/\mu$ is a class of η_1/μ that supports a subalgebra of $\mathbf{A}/\mu$ isomorphic to $\mathbf{F}$. Since η_1/μ is abelian, it follows that $\mathbf{F}$ is abelian too. □

Lemma 3.5. *If $\mathcal{V}$ is a nontrivial variety with a 0-ary function symbol, and all finitely generated members of $\mathcal{V}$ are free, then the nonconstant unary polynomial operations of $\mathbf{F}_{\mathcal{V}}(x)$ are injective.*

Proof. We first show that the nonconstant unary term operations act injectively on $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x)$. Here we use a symbol, say r, for both an element of F and also for a unary term operation $r^{\mathbf{F}}$ that represents the element r, i.e. $r = r^{\mathbf{F}}(x)$. If $r, s \in F$, we will use the notation rs for $r^{\mathbf{F}}(s)$. Thus, our goal is to show that if $r \in F \setminus \{0\}$, then $rs = rt$ implies $s = t$ for all $s, t \in F$.

Let $\eta_1, \eta_2, \Delta \in \mathbf{Con}(\mathbf{F}\times\mathbf{F})$ be the coordinate projection kernels and the congruence obtained from collapsing the diagonal. Suppose that $r, s, t \in F$, and that $rs = rt$ while $s \neq t$.

By Lemma 3.3 the element $(s,t) \in F \times F$ freely generates a subalgebra of $\mathbf{F} \times \mathbf{F}$ that is isomorphic to $\mathbf{F}_{\mathcal{V}}(1)$, hence it is a simple subalgebra that we denote by $\mathbf{T}$. Since $\mathbf{F}$ is abelian by Lemma 3.4, we have that (s,t) and $(0,0)$ are not Δ-related, so $\Delta|_T$ is trivial. But $rs = rt$ implies that $(rs, rt) \equiv_\Delta (0,0)$, so $(rs,rt) = (0,0)$. This shows that if $s \neq t$ and $rs = rt$, then $rs = 0 = rt$. At least one of s and t is not 0, and the situation between s and t has been symmetric up to this point, so assume that $s \neq 0$.

As before, the element (s,x) generates a simple subalgebra $\mathbf{X}$ of $\mathbf{F}\times\mathbf{F}$, since $x \neq 0$. The assumption $s \neq 0$ implies that (s,x) and $(0,0)$ are not η_1-related. Therefore $\eta_1|_X$ is trivial. But $(rs, rx) \equiv_{\eta_1} (0,0)$, so $(rs,rx) = (0,0)$. Hence $r = rx = 0$. This shows that our statement holds for unary term operations of $\mathbf{F}$.

Now we generalize our conclusion from unary term operations to unary polynomial operations of $\mathbf{F}$.

Assume that $p(x) = t^{\mathbf{F}}(x, \mathbf{u})$ for some term t and some tuple $\mathbf{u}$. If $p(a) = p(b)$, then

$$t^{\mathbf{F}}(a, \underline{\mathbf{u}}) = t^{\mathbf{F}}(b, \underline{\mathbf{u}}).$$

$\mathbf{F}$ is abelian by Lemma 3.4, therefore the last displayed equality is equivalent to

$$t^{\mathbf{F}}(a, \underline{\mathbf{0}}) = t^{\mathbf{F}}(b, \underline{\mathbf{0}}),$$

by the term condition. This shows that the unary polynomial $p(x) = t^{\mathbf{F}}(x, \mathbf{u})$ has the same kernel as the "twin" unary term operation $t^{\mathbf{F}}(x, \mathbf{0})$. But such kernels have

been shown to be trivial or universal in the first part of this proof, so they remain so here. I.e., any nonconstant unary polynomial operation acts injectively on $\mathbf{F}$. $\square$

Now we are prepared to prove the main result of this subsection.

Theorem 3.6. *Assume that $\mathcal{V}$ is a nontrivial variety such that the finitely generated algebras in $\mathcal{V}$ are free. If $\mathcal{V}$ has at least one 0-ary function symbol, then $\mathcal{V}$ is definitionally equivalent to either the variety of pointed sets or a variety of vector spaces over a division ring.*

Proof. Let $\mathcal{M}$ be a minimal subvariety of $\mathcal{V}$. By the same argument we used in Theorem 3.1, $\mathcal{M}$ also has the property that its finitely generated algebras are free. We first prove the theorem for $\mathcal{M}$, then lift the result to $\mathcal{V}$, as we did in Theorem 3.1.

All the lemmas proved for $\mathcal{V}$ in this subsection hold for $\mathcal{M}$. In particular,

(i) $\mathcal{M}$ has only one 0-ary function symbol, up to equivalence, which we denote by 0;
(ii) $\{0\}$ is the unique 1-element subalgebra in every member of $\mathcal{M}$, and
(iii) the unique finitely generated simple algebra in $\mathcal{M}$, up to isomorphism, is $\mathbf{F}_{\mathcal{M}}(1) = \mathbf{F}_{\mathcal{V}}(1)$.

By the minimality of $\mathcal{M}$, $\mathcal{M} = \mathsf{H}\,\mathsf{S}\,\mathsf{P}(\mathbf{F}_{\mathcal{M}}(1))$, and the free algebras of $\mathcal{M}$ therefore lie in $\mathsf{S}\,\mathsf{P}(\mathbf{F}_{\mathcal{M}}(1))$. This latter class contains all the free algebras of $\mathcal{M}$, hence contains all of the finitely generated members of $\mathcal{M}$, hence generates $\mathcal{M}$ as a universal class:

$$\mathcal{M} = \mathsf{S}\,\mathsf{P}_U(\mathsf{S}\,\mathsf{P}(\mathbf{F}_{\mathcal{M}}(1))) = \mathsf{S}\,\mathsf{P}\,\mathsf{P}_U(\mathbf{F}_{\mathcal{M}}(1)). \tag{3.2}$$

By Lemma 3.4, $\mathbf{F}_{\mathcal{M}}(1)$ is abelian, hence from (3.2) we deduce that $\mathcal{M}$ is an abelian variety.

As a first case, assume that $\mathcal{M}$ is affine. It follows from facts (i) and (ii) above and Lemma 4.3 of [11] that $\mathcal{M}$ is definitionally equivalent to a variety of left R-modules for some ring R. One realization of $\mathbf{F}_{\mathcal{M}}(1)$ has universe R, generator 1, and term operations of the form

$$r_1x_1 + \cdots + r_hx_h, \quad r_i \in R.$$

Each left ideal of R induces a congruence on this algebra. Since $\mathbf{F}_{\mathcal{M}}(1)$ is simple, R can have no nontrivial proper left ideals, hence R must be a division ring.

For the remaining case we may assume, from Theorem 2.4, that $\mathcal{M}$ has an affine obstruction $\mathbf{S}$ (see Definition 2.3). The element of S referred to as 0 in Definition 2.3 is a singleton subuniverse of $\mathbf{S}$, therefore fact (ii) ensures that it must be the element named by our constant symbol 0. It is easy to see that any nontrivial subalgebra of an affine obstruction $\mathbf{S}$ which contains 0 is again an affine obstruction (i.e., inherits properties (1)–(4) of Lemma 2.1). Since we know from Lemma 3.3 that every nontrivial 1-generated subalgebra of $\mathbf{S}$ is isomorphic to $\mathbf{F}_{\mathcal{M}}(1)$, we conclude that $\mathbf{F}_{\mathcal{M}}(1)$ has Property P.

Claim 3.7. $\mathbf{F}_{\mathcal{M}}(1)$ *has size* 2.

Proof of Claim. Assume otherwise that there are distinct nonzero elements a, b in $F_{\mathcal{M}}(1)$. The congruence $\mathrm{Cg}(a, b)$ is nontrivial, hence by the simplicity of $\mathbf{F}_{\mathcal{M}}(1)$ there is a unary polynomial $p(x)$ of $\mathbf{F}_{\mathcal{M}}(1)$ such that $p(a) = 0 \neq p(b)$, or the same with a and b interchanged. But $p(a) = 0$ implies $p(0) = 0$, by Property P, showing that $(a, 0)$ is a nontrivial pair in $\ker(p)$. On the other hand (a, b) is a pair not in $\ker(p)$. This contradicts Lemma 3.5, which establishes that unary polynomials of $\mathbf{F}_{\mathcal{M}}(1)$ are constant or injective. ∎

Claim 3.7, together with earlier information, yields that $\mathbf{F}_{\mathcal{M}}(1)$ is a 2-element, nonaffine, abelian algebra with a singleton subalgebra named by a constant. There is one such algebra up to definitional equivalence, namely the 2-element pointed set. (The simplest way to affirm this is to refer to Post's classification of 2-element algebras, but one doesn't need a result of such depth to make this conclusion.)

Since $\mathcal{M}$ is generated by $\mathbf{F}_{\mathcal{M}}(1)$, which is equivalent to a pointed set, it follows that $\mathcal{M}$ is definitionally equivalent to the variety of pointed sets in the case we are considering.

We have shown that $\mathcal{M}$ is definitionally equivalent to a variety of vector spaces over a division ring or the variety of pointed sets. We now argue that $\mathcal{V} = \mathcal{M}$ using the same type of argument used in Theorem 3.1.

If $\mathcal{V} \neq \mathcal{M}$, there is a finitely generated algebra in $\mathcal{V} \setminus \mathcal{M}$, which we may assume is $\mathbf{A} := \mathbf{F}_{\mathcal{V}}(m)$. Then $\mathbf{A}$ has an m-element generating set that is minimal under inclusion as a generating set. Let $\mathbf{B}$ be the m-generated free algebra in $\mathcal{M}$. The algebra $\mathbf{B}$ also has an m-element minimal generating set. But $\mathbf{B} \in \mathcal{M}$, so $\mathbf{B} \in \mathcal{V}$, and $\mathbf{B}$ cannot be isomorphic to $\mathbf{A}$, so $\mathbf{B} \cong \mathbf{F}_{\mathcal{V}}(n)$ for some $n \neq m$. This implies that $\mathbf{B}$ has an n-element minimal generating set as well as an m-element minimal generating set. But there does not exist a vector space nor a pointed set that has minimal generating sets of different cardinalities. We conclude that $\mathcal{V} = \mathcal{M}$. □

4. Discussion

Throughout this paper our arguments depended on some strong but odd assumptions, namely that a 1-element $\mathcal{V}$-algebra is free and that a finitely generated simple $\mathcal{V}$-algebra is free. One might wonder whether anything can be proved for varieties where only the "large" finitely generated algebras are assumed to be free. Specifically, one might ask what can be said about the varieties $\mathcal{V}$ satisfying the following property: There exists a natural number k such that every finitely generated algebra in $\mathcal{V}$ is either free or can be generated by $\leq k$ elements.

Unfortunately there is a seemingly-unclassifiable collection of varieties for which $\mathbf{F}_{\mathcal{V}}(j) \cong \mathbf{F}_{\mathcal{V}}(k)$ for some $j < k$. For any given $j < k$ the varieties with this property represent a filter in the lattice of interpretability types. In such varieties every finitely generated algebra can be generated by $\leq k$ elements, so the conditions of the question

are satisfied. This suggests that there is no nice classification of the varieties $\mathcal{V}$ satisfying the property above.

However, if we restrict our attention to locally finite varieties, then we can prove the following.

Theorem 4.1. *Let $\mathcal{V}$ be a nontrivial locally finite variety. If there exists a natural number k such that every finitely generated algebra $\mathcal{V}$ is either free or can be generated by $\leq k$ elements, then every nonsingleton algebra in $\mathcal{V}$ is free. In fact, $\mathcal{V}$ is definitionally equivalent to*

1. *the variety of sets,*
2. *the variety of pointed sets,*
3. *a variety of vector spaces over a finite field, or*
4. *a variety of affine spaces over a finite field.*

Caveat: While in the earlier part of the paper our "pointed sets" and "vector spaces" each had a (unique) 0-ary term operation, in this theorem we allow the constants of the algebras in cases (2) and (3) to be constant 0-ary term operations *or* constant 1-ary term operations. If these constants are 1-ary term operations and there are no constant 0-ary term operations, then no 1-element algebra of the variety is free, but all the other algebras are free.

Proof. First observe that any variety satisfying the hypotheses of the theorem must be a minimal variety. For if $\mathcal{M}$ is a minimal subvariety of $\mathcal{V}$, then the sequence $(\mathbf{F}_{\mathcal{M}}(p))_{p\in\omega}$ consists of algebras in $\mathcal{V}$ whose sizes increase with p, and which require more generators as p increases. It follows from the hypotheses of the theorem that some tail end of this sequence is cofinal in the sequence $(\mathbf{F}_{\mathcal{V}}(q))_{q\in\omega}$. Hence the algebras in the first sequence generate the same variety as the algebras in the second sequence, i.e. $\mathcal{M} = \mathcal{V}$.

By local finiteness, the hypotheses on $\mathcal{V}$ ensure that there are at most finitely many (say C) isomorphism types of finitely generated non-free algebras in $\mathcal{V}$. Local finiteness ensures that $\mathbf{F}_{\mathcal{V}}(n)$ cannot be m-generated if $m < n$. Hence if $n \geq C$, it follows that there are $n+1$ free algebras that can be generated by $\leq n$ elements $(\mathbf{F}_{\mathcal{V}}(0), \ldots, \mathbf{F}_{\mathcal{V}}(n))$ and C non-free algebras that can be generated by $\leq n$ elements, hence a total of $n+C$ algebras in $\mathcal{V}$ that can be generated by $\leq n$ elements. This says precisely that the *G-spectrum* of $\mathcal{V}$ satisfies $G_{\mathcal{V}}(n) = n + C$ whenever $n \geq C$. (The G-spectrum of a locally finite variety $\mathcal{V}$ is the function whose value at n is the number of isomorphism types of algebras in $\mathcal{V}$ that can be generated by $\leq n$ elements.)

It is known that a locally finite variety $\mathcal{V}$ whose G-spectrum $G_{\mathcal{V}}(n)$ is bounded above by a polynomial function of n must be abelian ([6], Theorem 8.15). So at this point we know that our variety $\mathcal{V}$ is a locally finite, minimal, abelian variety. These have been classified in [9, 12, 13]. Such varieties are definitionally equivalent to either a matrix power of the variety of sets, a matrix power of a variety of pointed sets (note

the caveat between the theorem statement and the start of the proof), or to an affine variety over a finite simple ring where each member has a singleton subuniverse. We will complete the proof of the theorem by examining the clones of such algebras.

Let $\mathbf{S}$ be a strictly simple generator of our locally finite, minimal, abelian variety $\mathcal{V}$. It follows from the results in [9, 12, 13] that $\mathbf{S}$ is isomorphic to an algebra that is term equivalent to (i.e., has the same underlying set and the same non-nullary term operations as) one of the following algebras:

(i) $\mathbf{A} = (\mathbf{2};\emptyset)^{[d]}$ $(d \geq 1)$, the d-th matrix power of the 2-element set $\mathbf{2} = \{0, 1\}$;
(ii) $\mathbf{A} = (\mathbf{2}; 0)^{[d]}$ $(d \geq 1)$, (the d-th matrix power of the 2-element pointed set $(\mathbf{2}; 0)$;
(iii) an affine reduct $\mathbf{A}$ of a finite simple module $\mathbf{M}$ such that $\mathbf{A}$ has the same ring as $\mathbf{M}$.

In each one of these cases, the fact that $\mathbf{S}$ generates $\mathcal{V}$ implies that

$$|\mathbf{F}_{\mathcal{V}}(n)| = |\mathrm{Clo}_n(\mathbf{A})| \quad \text{for every } n \geq 1,$$

where $\mathrm{Clo}_n(\mathbf{A})$ denotes the set of n-ary term operations of $\mathbf{A}$ (the n-ary sort of the clone of $\mathbf{A}$). Thus, if $\mathcal{V}$ satisfies the assumptions of the theorem, then the (increasing) sequence of all sizes of finite algebras in $\mathcal{V}$ must have the same tail end as the sequence $(|\mathrm{Clo}_n(\mathbf{A})|)_{0<n<\omega}$. To finish the proof of the theorem, we have to deduce from this condition that

- $d = 1$ in cases (i) and (ii), and
- $\mathbf{A}$ is a 1-dimensional vector space or affine space over a finite field in case (iii).

Cases (i)–(ii). Every operation $f \in \mathrm{Clo}_n(\mathbf{A})$ has the form

$$\begin{aligned} &f\colon (\mathbf{2}^d)^n \to \mathbf{2}^d,\\ &\quad \big((x_{0,0}, \dots, x_{0,d-1}), \dots, (x_{n-1,0}, \dots, x_{n-1,d-1})\big) \mapsto \big(f_0(x_{i_0,j_0}), \dots, f_{d-1}(x_{i_{d-1},j_{d-1}})\big) \end{aligned}$$

where, for each ℓ, either $f_\ell = id$ and (i_ℓ, j_ℓ) is a pair of integers with $0 \leq i_\ell < n$, $0 \leq j_\ell < d$, or we are in case (ii) and f_ℓ is the (unary) constant operation with value 0 and the pair (i_ℓ, j_ℓ) is irrelevant. It is easy to check that different choices yield different operations. Hence $|\mathrm{Clo}_n(\mathbf{A})| = (nd)^d$ in case (i) and $|\mathrm{Clo}_n(\mathbf{A})| = (nd+1)^d$ in case (ii).

For every finite set B with $0 \in B$, the algebra $(B;\emptyset)^{[d]}$ belongs to the variety generated by $(\mathbf{2};\emptyset)^{[d]}$, and the algebra $(B; 0)^{[d]}$ belongs to the variety generated by $(\mathbf{2}; 0)^{[d]}$. Hence, $\mathcal{V}$ contains algebras of sizes m^d for every $m \geq 1$. Since our assumptions force that the (increasing) sequence of all sizes of finite algebras in $\mathcal{V}$ has the same tail end as the sequence $(|\mathrm{Clo}_n(\mathbf{A})|)_{0<n<\omega}$, we get that a tail end of the sequence $(m^d)_{0<m<\omega}$ must be a subsequence of a tail end of the sequence $\big((nd)^d\big)_{0<n<\omega}$ or $\big((nd+1)^d\big)_{0<n<\omega}$, according to whether we are in case (i) or (ii). It is easy to see that in both cases this will hold only if $d = 1$.

Case (iii). Let $\mathbf{A}$ be an affine reduct of a finite, simple R-module such that the ring of $\mathbf{A}$ is also R. Since we are only interested in the term operations of $\mathbf{A}$, we may assume without loss of generality that R and $\mathbf{M}$ are unital and $\mathbf{M}$ is a faithful R-module. Since $\mathbf{M}$ is finite and simple, it follows that there exist a finite field K and a positive integer d such that R is the ring of $d \times d$ matrices with entries in K, and $\mathbf{M}$ is a d-dimensional K-vector space with the usual action of R as an R-module.

Since $\mathbf{A}$ is an affine reduct of $\mathbf{M}$ with the same ring R as $\mathbf{M}$, Lemma 4.3 of [12] implies that there exists a left ideal L of R such that

$$\text{(4.1)}\quad \mathrm{Clo}_n(\mathbf{A}) = \left\{ \sum_{i=0}^{n-1} r_i x_i : r_0, \dots, r_{n-1} \in R \ \text{ and } \ 1 - \sum_{i=0}^{n-1} r_i \in L \right\} \quad \text{for all } n \geq 1.$$

Thus, $|\mathrm{Clo}_n(\mathbf{A})| = |R|^{n-1}|L| = |M|^{d(n-1)}|L| = |A|^{d(n-1)}|L|$ for all $n \geq 1$. The variety $\mathcal{V}$ contains finite algebras of sizes $|A|^m = |S|^m$ for every $m \geq 1$. Now, if $d > 1$, then no tail end of the sequence $(|A|^m)_{0<m<\omega}$ is a subsequence of any tail end of the sequence $(|\mathrm{Clo}_n(\mathbf{A})|)_{0<n<\omega} = (|A|^{d(n-1)}|L|)_{0<n<\omega}$. Therefore we conclude the same way as before that $d = 1$. This implies that $R = K$ and $\mathbf{M}$ is a 1-dimensional K-vector space. Hence, either $L = K$ or $L = \{0\}$, which implies by (4.1) that $\mathbf{A}$ is term equivalent to either the vector space $\mathbf{M}$, or the corresponding affine space (i.e., the full idempotent reduct of $\mathbf{M}$). $\square$

Now we turn to the opposite type of question: what can one say about the varieties for which there is a natural number n such that every $(\leq n)$-generated algebra is free? If n is large enough, must all algebras in the variety be free? We show that the answer to this is negative for any natural number n.

Theorem 4.2. *For any natural number n there exists a variety with the property that every $(\leq n)$-generated algebra is free, but some $(n+1)$-generated algebra in the variety is not free.*

Proof. An $(m+1)$-ary (first variable) *semiprojection* on a set A is an $(m+1)$-ary operation $s(x_0, x_1, \dots, x_m)$ on A such that for any $\mathbf{a} \in A^{m+1}$ we have

$$s(a_0, a_1, \dots, a_m) = a_0$$

whenever $a_i = a_j$ for some $i \neq j$. This property can be expressed by identities, so starting with any variety $\mathcal{V}$ we can add an $(m+1)$-ary function symbol s to the language and define $\mathcal{V}_s$ to be the variety of all $\mathcal{V}$-algebras expanded by an $(m+1)$-ary (first variable) semiprojection.

The added semiprojection operation acts like first projection on any algebra in $\mathcal{V}_s$ that has cardinality at most m. Hence any algebra of size at most m in $\mathcal{V}_s$ is definitionally equivalent to an algebra in $\mathcal{V}$.

If $\mathcal{V}$ is the variety of sets, then this construction with $m = n$ yields a variety $\mathcal{V}_s$ in which every algebra that is generated by at most n elements will be definitionally

equivalent to a set, hence will be free. Now let $\mathbf{B}$ be the $(n+1)$-element algebra in $\mathcal{V}_s$ where s interprets as a first projection, so $\mathbf{B}$ is definitionally equivalent to a set. This algebra is not free, because there exist $(n+1)$-generated algebras in $\mathcal{V}_s$ that are not homomorphic images of $\mathbf{B}$. For example, any $(n+1)$-element algebra $\mathbf{A}$ in $\mathcal{V}_s$ where s is a (first variable) semiprojection other than a projection has this property.

Similarly, if $\mathcal{V}$ is the variety of vector spaces over the 2-element field, and we let $m = 2^n$, then the (2^n+1)-ary semiprojection s acts like first projection on any algebra in $\mathcal{V}_s$ generated by at most n elements. Again, all algebras in $\mathcal{V}_s$ that are generated by at most n elements will be free, but there will be $(n+1)$-generated algebras in $\mathcal{V}_s$ that are not free. $\square$

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